

## MATHEMATICS

# Cognitive Supports for Analogies in the Mathematics Classroom

Lindsey E. Richland,<sup>1\*</sup> Osnat Zur,<sup>2</sup> Keith J. Holyoak<sup>2</sup>

Mathematics education is a critical concern worldwide. Within the United States, the mathematics educational system needs improvement (1, 2). By the time U.S. students reach middle school, they have fallen below their international peers on assessments of mathematics performance (3, 4). Failures in students' mathematics learning reduce high-school retention and become formidable barriers to college admissions and entry into math and science careers. Many factors contribute to mathematics achievement. We investigated how certain mathematics classroom activities differ between the United States and nations in which students score higher on international tests (3, 4). We focused on factors of cognition and memory, which can be distinguished from cultural differences in instruction.

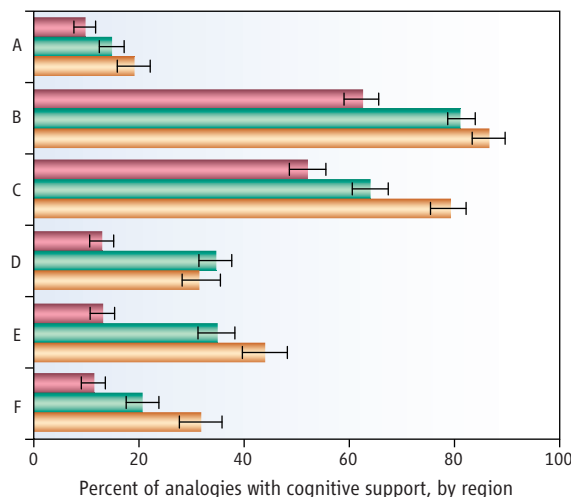
The video portion of the Trends in International Mathematics and Science Study (TIMSS 1999 Video Study), a large-scale international video study of classroom mathematics instruction, indicated that American teachers introduced conceptually connected, rich problems at rates similar to teachers from higher-achieving countries. However, they engaged students in complex connected reasoning and problem-solving substantially less often (5). One sophisticated reasoning practice available to children is the use of analogy and similar relational comparisons, which promote flexible conceptual learning and problem-solving (6). Analogy allows students to use commonalities between mathematical representations to help understand new problems or concepts, thereby contributing to integral components of mathematical proficiency (1, 7).

Learning by analogy typically involves finding a set of systematic correspondences (a mapping) between a better-known source analog and a more novel target. The source and the target can be within a single domain (e.g., solving inequalities is like

solving equations) or across domains (e.g., balancing equations is like balancing a scale) (8, 9). Mathematical reasoning involves understanding abstract relations (such as equality, proportion, and integral) that can appear in different contexts (7, 10). Such abstract relations may be best taught by drawing parallels between similar examples (9, 11–13). Even so, children and novices often fail to notice or benefit from such instructional comparisons (9, 14, 15) when they are presented without supportive cues, such as hints, prompting questions, or elaborations of the analogy (9, 12, 16, 17).

Mathematics teachers in the United States commonly introduce analogy-based instruction in their lessons, but not always in ways that encourage active reasoning by the students (18, 19). We analyzed the ways that analogies are used in U.S. classrooms compared with two high-achieving regions in Asia: Hong Kong (Special Administrative Region, China) and Japan. All instances of relational comparisons (analogies) were identified in 10 eighth-grade lessons from different teachers videotaped in each country, randomly sampled from the TIMSS 1999 Video database (5). Hong Kong and Japan were selected for comparison to the United States because their students consistently outperform U.S. students on the TIMSS International achievement tests (3, 4). In addition, their classroom instructional practices are very different from each other (5). Each relational comparison was then analyzed using qualitative codes to gather quantitative data about these reasoning events. Based on techniques developed in previous video surveys, codes were developed in an iterative strategy by alternating between the research literature and observations of the classrooms (5, 20). Intercoder reliability was calculated between coders and the first two authors.

Variations in the effective use of analogies in math instruction across countries may contribute to performance differences in the TIMSS studies.



**Frequency of cognitive support for analogies.** Teachers reinforced a percentage of their analogies (bars) by providing cognitive supports (labeled A through F; defined in the text). Supports were provided less frequently by U.S. teachers (red) than by teachers from Hong Kong (green) or Japan (orange). Error bars represent standard error per country per support code.

Coders were native to the three videotaped countries and were psychologically naïve, although they were not blind to the country identity because video data precluded this possibility (21).

Codes identified teaching strategies that did or did not exhibit the use of sound cognitive principles for supporting relational learning, identified in laboratory studies over the past three decades. Part of the novelty of this research was to translate well-established principles of relational learning into codable behaviors relevant to classroom instruction in mathematics and across cultures. Our codes were motivated by the body of basic research on relational learning, imagery, gesture, and working memory (9, 11, 19, 22–27).

Codes fell into two categories: those measuring characteristics of the source (e.g., using a scale as a familiar source to teach about balancing equations) and those measuring properties that increased the vividness of the alignment and mapping of the analogy as a whole (e.g., using a real scale while practicing balancing equations). These codes captured strategies that reduce processing demands on retrieval and working

<sup>1</sup>Department of Education, University of California, Irvine, CA 92697, USA. <sup>2</sup>Department of Psychology, University of California, Los Angeles, CA 90095, USA.

\*Author for correspondence. E-mail: l.e.richland@uci.edu

memory and that draw attention to alignment of relations—all of which are aids to learning in laboratory studies of analogy and transfer. For example, children show greater transfer when the source is relatively familiar, as in the case of a scale and balancing of equations (26, 27). Augmenting the source with visual representations such as a diagram can also increase transfer (9, 28). Relative to auditory presentation, a visual display normally persists over time, reducing demands on working memory (25). Providing spatial cues such as position and arrows (11, 25), or comparative gestures (22, 23, 29) can serve to highlight correspondences.

Adherence to six principles was coded (see figure). Three of the principles concerned the teachers' sources: The teachers (A) used a familiar source analog to compare to the target analog being taught; (B) presented the source analog visually; and (C) kept the source visible to learners during comparison with the target. Other principles served to enhance the vividness of the relational comparison used: Teachers (D) used spatial cues to highlight the alignment between corresponding elements of the source and target (e.g., diagramming a scale below the equal sign of an equation); (E) used hand or arm gestures that signaled an intended comparison (e.g., pointing back and forth between a scale and an equation); and (F) used mental imagery or visualizations (e.g., "picture a scale when you balance an equation"). Principles B to F can be viewed as special cases of the overarching principle that appropriate visual and spatial cues aid comprehension of abstract relations (11, 22, 23, 25, 28, 29).

Teachers in all three countries produced numerous relational comparisons during the 10 eighth-grade mathematics lessons. Every lesson contained relational comparisons. A total of 195 units were identified in the U.S. lessons (mean of 20, range of 9 to 30 per lesson), 185 were identified in Hong Kong lessons (mean of 18, range of 7 to 27 per lesson), and 139 were identified in Japanese lessons (mean of 14, range of 9 to 25 per lesson).

National differences emerged in adherence to sound cognitive principles for teaching by relational comparisons. For all six principles that we coded, the U.S. sample yielded lower scores, indicating less promotion of relational learning, than did either of the Asian samples (see figure) (30). For example, teachers in both Asian regions used spatial supports for comparison more than twice as often as did their U.S. counter-

parts. These teachers also used far more gestures that emphasized comparison than did U.S. teachers, even though the latter used gestures of some kind almost equally often. Hong Kong teachers were almost twice as likely to prompt mental and visual imagery as were U.S. teachers, and Japanese teachers were even more likely.

This "teaching gap" may reflect different cultural orientations to relational reasoning. Hong Kong and Japanese teachers appear to be more attentive to the processing demands of relational comparisons than are U.S. teachers. Their teaching reflects the use of strategies to reduce processing demands on their students. Such differences in adherence to sound cognitive principles may have a real impact on the likelihood that students benefit from analogies as instructional tools. If the source analog is not familiar and not visible, then students may struggle with processing. First, students will need to perform a taxing memory search to understand the source. Then, assuming that memory retrieval is successful, lack of visual availability will place further burdens on working memory during production of the relational comparison. Finally, lack of supporting cues to guide the comparison itself may result in the student learning much less than, or something quite different from, the new relational concept the teacher means to convey. Unsuccessful analogies may produce misunderstandings that can even lead to harmful misconceptions (12, 31).

These cross-national differences in teaching practices suggest ways in which American mathematics education might be improved by building on existing practices. Relative to nations in which students achieve high TIMSS scores, U.S. mathematics educators introduce a similar number of analogies but offer less in terms of cognitive backup to help their students benefit from these analogies. Findings fit an emerging pattern: U.S. teachers provide high-quality learning opportunities to their students but provide less of the support that would enable their students to reap maximal benefits (32).

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#### Supporting Online Material

[www.sciencemag.org/cgi/content/full/316/5828/1128/DC1](http://www.sciencemag.org/cgi/content/full/316/5828/1128/DC1)

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